

ON THE OPTIMAL CONSTANTS IN KORN'S AND GEOMETRIC RIGIDITY ESTIMATES, IN BOUNDED AND UNBOUNDED DOMAINS, UNDER NEUMANN BOUNDARY CONDITIONS

MARTA LEWICKA AND STEFAN MÜLLER

ABSTRACT. We are concerned with the optimal constants: in the Korn inequality under tangential boundary conditions on bounded sets $\Omega \subset \mathbb{R}^n$, and in the geometric rigidity estimate on the whole \mathbb{R}^2 . We prove that the latter constant equals $\sqrt{2}$, and we discuss the relation of the former constants with the optimal Korn's constants under Dirichlet boundary conditions, and in the whole \mathbb{R}^n , which are well known to equal $\sqrt{2}$. We also discuss the attainability of these constants and the structure of deformations/displacement fields in the optimal sets.

1. INTRODUCTION AND THE MAIN RESULTS

In this paper we are concerned with the optimal constants in the Korn inequality [10, 11] and in the Friesecke-James-Müller geometric rigidity estimate [7, 8].

Let Ω be an open, bounded, and connected subset of \mathbb{R}^n with Lipschitz continuous boundary. The Korn inequality [10, 11, 13] states that there exists a constant $C(\Omega)$ depending only on Ω , such that for all $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ there holds:

$$(1.1) \quad \min_{A \in so(n)} \|\nabla u - A\|_{L^2(\Omega)} \leq C(\Omega) \|D(u)\|_{L^2(\Omega)},$$

where by $D(u) = \frac{1}{2} (\nabla u + (\nabla u)^T)$ we mean the symmetric part of ∇u .

Let now \vec{n} denote the outward unit normal on $\partial\Omega$. Given (1.1), it is not hard to deduce (see Lemma 2.1) the following version of Korn's inequality subject to tangential boundary conditions. Namely, there exists a constant $\kappa(\Omega)$, depending only on Ω , such that for all $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ satisfying $u \cdot \vec{n} = 0$ on $\partial\Omega$ there holds:

$$(1.2) \quad \min_{A \in L_\Omega} \|\nabla u - A\|_{L^2(\Omega)} \leq \kappa(\Omega) \|D(u)\|_{L^2(\Omega)},$$

where by L_Ω above we denote the linear space of skew-symmetric matrices that are gradients of affine maps tangential on the boundary of Ω :

$$L_\Omega = \{A \in so(n); \exists a \in \mathbb{R}^n \quad \forall x \in \partial\Omega \quad (Ax + a) \cdot \vec{n}(x) = 0\}.$$

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The optimal constant in (1.2) is given by:

$$(1.3) \quad \kappa(\Omega) = \sup \left\{ \min_{A \in L_\Omega} \|\nabla u - A\|_{L^2(\Omega)}; u \in W^{1,2}(\Omega, \mathbb{R}^n), u \cdot \vec{n} = 0 \text{ on } \partial\Omega \right. \\ \left. \text{and } \|D(u)\|_{L^2(\Omega)} = 1 \right\},$$

and we aim to study its relation to Korn's constant in the whole \mathbb{R}^n , which is $\sqrt{2}$ (see Lemma 2.2):

$$(1.4) \quad \kappa(\mathbb{R}^n) = \sup \left\{ \|\nabla u\|_{L^2(\mathbb{R}^n)}; u \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^n), \|D(u)\|_{L^2(\mathbb{R}^n)} = 1 \right\} = \sqrt{2}.$$

In this setting, our first set of main results is:

Theorem 1.1. *For any open, bounded, Lipschitz, connected $\Omega \subset \mathbb{R}^n$:*

$$(1.5) \quad \kappa(\Omega) \geq \kappa(\mathbb{R}^n) = \sqrt{2}.$$

In fact, $\kappa(\Omega)$ may be arbitrarily large. In Example 3.3 we will recall our construction in [12] which implies that for a sequence of thin shells around a sphere, the Korn constants go to ∞ as the thickness goes to 0. On the other hand, as we show in Example 3.2, there is: $\kappa([0, 1]^2) = \sqrt{2}$.¹ We however have:

Theorem 1.2. *Assume that there exists a sequence $\{u_k\}_{k=1}^\infty$, $u_k \in W^{1,2}(\Omega, \mathbb{R}^n)$, $u_k \cdot \vec{n} = 0$ on $\partial\Omega$, with the following properties:*

- (i) u_k converges to 0 weakly in $W^{1,2}(\Omega, \mathbb{R}^n)$,
- (ii) $\|D(u_k)\|_{L^2(\Omega)} = 1$,
- (iii) $\lim_{k \rightarrow \infty} \|\nabla u_k\|_{L^2(\Omega)} = \kappa(\Omega)$.

Then $\kappa(\Omega) = \sqrt{2}$.

Theorem 1.3. *If $\kappa(\Omega) > \sqrt{2}$ then the supremum in the definition (1.3) is attained. More precisely, for every $A_0 \in L_\Omega$ there exists $u \in W^{1,2}(\Omega, \mathbb{R}^n)$, such that $u \cdot \vec{n} = 0$ on $\partial\Omega$, $D(u) \neq 0$, and:*

$$(1.6) \quad \min_{A \in so(n)} \|\nabla u - A\|_{L^2(\Omega)} = \|\nabla u - A_0\|_{L^2(\Omega)} = \kappa(\Omega) \|D(u)\|_{L^2(\Omega)}.$$

Theorem 1.4. *The vector fields u for which Korn's constant $\kappa(\Omega)$ is attained:*

$$(1.7) \quad \left\{ u \in W^{1,2}(\Omega, \mathbb{R}^n); u \cdot \vec{n} = 0 \text{ on } \partial\Omega, u \text{ satisfies (1.6) for some } A_0 \in L_\Omega \right\};$$

form a closed linear subspace of $W^{1,2}(\Omega, \mathbb{R}^n)$. Moreover, if $\kappa(\Omega) > \sqrt{2}$ then this space is of finite dimension.

In the second part of this paper we concentrate on the nonlinear version of Korn's inequality, namely the Friesecke-James-Müller geometric rigidity estimate [7, 8]. It states that for an open, bounded, smooth and connected domain $\Omega \subset \mathbb{R}^n$, there exists a constant $\kappa_{nl}(\Omega)$ depending only on Ω , such that for every $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ there holds:

$$(1.8) \quad \min_{R \in SO(n)} \|\nabla u - R\|_{L^2(\Omega)} \leq \kappa_{nl}(\Omega) \|\text{dist}(\nabla u, SO(n))\|_{L^2(\Omega)}.$$

¹We are able to prove that for smooth domains there always holds: $\kappa(\Omega) > \sqrt{2}$. The proof of this fact will appear elsewhere.

Define:

$$(1.9) \quad \kappa_{nl}(\mathbb{R}^n) = \sup \left\{ \min_{R \in SO(n)} \frac{\|\nabla u - R\|_{L^2(\mathbb{R}^n)}}{\|\text{dist}(\nabla u, SO(n))\|_{L^2(\mathbb{R}^n)}}; u \in W_{loc}^{1,2}(\mathbb{R}^n, \mathbb{R}^n), \right. \\ \left. \text{dist}(\nabla u, SO(n)) \in L^2(\mathbb{R}^n) \setminus \{0\} \right\}.$$

Our results in this context are restricted to dimension 2:

Theorem 1.5. *We have: $\kappa_{nl}(\mathbb{R}^2) = \sqrt{2}$. In particular:*

$$\forall u \in W_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}^2) \quad \text{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2) \implies \\ \min_{R \in SO(2)} \|\nabla u - R\|_{L^2(\mathbb{R}^2)} \leq \sqrt{2} \|\text{dist}(\nabla u, SO(2))\|_{L^2(\mathbb{R}^2)}.$$

Theorem 1.6. *For every rotation $R_0 \in SO(2)$ there exists $u \in W_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2) \setminus \{0\}$ such that:*

$$(1.10) \quad \min_{R \in SO(2)} \|\nabla u - R\|_{L^2(\mathbb{R}^2)} = \|\nabla u - R_0\|_{L^2(\mathbb{R}^2)} \\ = \sqrt{2} \|\text{dist}(\nabla u, SO(2))\|_{L^2(\mathbb{R}^2)}.$$

Theorem 1.7. *The vector fields for which the nonlinear Korn constant in (1.9) is attained, namely:*

$$\left\{ u \in W_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}^2); \text{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2), \right. \\ \left. u \text{ satisfies (1.10) for some } R_0 \in SO(2) \right\}$$

have the defining property that their gradients are of the form:

$$(1.11) \quad \nabla u(x) = R_0 R(\alpha(x)) + \begin{bmatrix} a(x) & b(x) \\ b(x) & -a(x) \end{bmatrix} \quad \text{with } R(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix},$$

for some $\alpha, a, b \in L^2(\mathbb{R}^2)$. Conversely, for every $\alpha \in L^2(\mathbb{R}^2)$ there exists $a, b \in L^2(\mathbb{R}^2)$ and $u \in W_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ such that (1.10) and (1.11) hold.

The proofs of the three Theorems above are independent from the proof of (1.8) in [7]. They rely on the conformal-anticonformal decomposition of 2×2 matrices, and it is not clear how this construction and methods could be extended to yield a result in higher dimensions $n > 2$.

There is an extensive literature relating to Korn's inequality and its applications, notably in linear elasticity [2, 3, 10, 13]. On the other hand, the nonlinear estimate (1.8) plays crucial role in models in nonlinear elasticity [8, 7]. Indeed, the relation between these two estimates is clear if we recall that the tangent space to $SO(n)$ at Id is $so(n)$. The blow-up rate and properties of $\kappa(\Omega)$ for thin spherical-like domains around a given surface were studied in [12]. The relations of $\kappa(\Omega)$ with the measure of axisymmetry of Ω have been discussed in [5]. An interesting extension of both Korn's and the geometric rigidity estimates under mixed growth conditions has been recently established in [4].

2. PRELIMINARIES

Recall that the linear space of skew-symmetric matrices is:

$$so(n) = \{A \in \mathbb{R}^{n \times n}; A = -A^T\}$$

while $SO(n)$ stands for the group of proper rotations:

$$SO(n) = \{R \in \mathbb{R}^{n \times n}; R^T = R^{-1} \text{ and } \det R = 1\}.$$

The scalar product and the (Frobenius) norm in the space of $n \times n$ (real) matrices $\mathbb{R}^{n \times n}$ are given by:

$$A : B = \text{tr}(A^T B) \quad |A|^2 = A : A.$$

We first notice the following characterization of the minimiser in (1.2):

Lemma 2.1. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^n)$, $u \cdot \vec{n} = 0$ on $\partial\Omega$. Then the minimum in the left hand side of (1.2) is attained, uniquely, at:*

$$A_0 = \mathbb{P}_{L_\Omega} \int_{\Omega} \nabla u,$$

where \mathbb{P}_{L_Ω} denotes the orthogonal projection of $\mathbb{R}^{n \times n}$ on L_Ω .

Proof. Let $A_0 \in L_\Omega$ be a minimiser of $\|\nabla u - A\|_{L^2(\Omega)}^2$ over L_Ω . Taking the derivative in the direction of $A \in L_\Omega$, one obtains:

$$\forall A \in L_\Omega \quad \int_{\Omega} (\nabla u - A_0) : A = 0.$$

Equivalently, there holds:

$$\left(\int_{\Omega} \nabla u - A_0 \right) \in L_\Omega^\perp,$$

which implies the lemma. ■

For convenience of the reader, we now sketch the proof of (1.4).

Lemma 2.2. *For every open, Lipschitz, connected $\Omega \subset \mathbb{R}^n$, the Korn constant under Dirichlet boundary conditions equals $\sqrt{2}$:*

$$(2.1) \quad \kappa_0(\Omega) = \sup \left\{ \|\nabla u\|_{L^2(\Omega)}; u \in W_0^{1,2}(\Omega, \mathbb{R}^n), \|D(u)\|_{L^2(\Omega)} = 1 \right\} = \sqrt{2}.$$

Proof. For every $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ we have:

$$(2.2) \quad \begin{aligned} 2 \int_{\Omega} |D(u)|^2 &= \int_{\Omega} |\nabla u|^2 + \int_{\Omega} \nabla u : (\nabla u)^T = \int_{\Omega} |\nabla u|^2 + \int_{\Omega} \text{tr}(\nabla u)^2 \\ &= \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\text{div } u|^2 + \int_{\Omega} (\text{tr}(\nabla u)^2 - (\text{tr} \nabla u)^2). \end{aligned}$$

When, additionally, Ω is bounded and $u \in W_0^{1,2}(\Omega, \mathbb{R}^n)$, this implies that:

$$2 \int_{\Omega} |D(u)|^2 = \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\text{div } u|^2,$$

because $(\text{tr}(\nabla u)^2 - (\text{tr} \nabla u)^2)$ is a null-Lagrangian, i.e. its integral depends only on the boundary value of u on $\partial\Omega$. We therefore conclude that, in this case: $\|\nabla u\|_{L^2(\Omega)} \leq \sqrt{2} \|D(u)\|_{L^2(\Omega)}$. The same inequality is also true on unbounded domains, because of the density of $C_c^\infty(\Omega, \mathbb{R}^n)$ in $W_0^{1,2}(\Omega, \mathbb{R}^n)$.

To prove that $\sqrt{2}$ is optimal and that it is attained, it is enough to take $u \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div} u = 0$ (when $n = 3$, take $u = \operatorname{curl} v$ for any compactly supported v). This achieves the proof. \blacksquare

We now recall the Poincaré inequality for tangential vector fields. The proof, which can be found in [1], is deduced through a standard argument by contradiction.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, Lipschitz set. For every $u \in W^{1,2}(\Omega, \mathbb{R}^n)$, $u \cdot \vec{n} = 0$ on $\partial\Omega$, there holds:*

$$(2.3) \quad \|u\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla u\|_{L^2(\Omega)},$$

where the constant $C(\Omega)$ depends only on Ω (it is independent of u).

3. THE OPTIMAL KORN CONSTANT $\kappa(\Omega)$: A PROOF OF THEOREM 1.1 AND TWO EXAMPLES

In the course of proof of Theorem 1.1, we will use the following observation:

Proposition 3.1. *For any $f \in L^2(\mathbb{R}^n)$ there holds:*

$$\lim_{R \rightarrow \infty} R^{-n/2} \|f\|_{L^1(B_R)} = 0,$$

on the ball $B_R = \{x \in \mathbb{R}^n, |x| \leq R\}$.

Proof. Fix $\epsilon > 0$. For m sufficiently large, one has $\|f\|_{L^2(\mathbb{R}^n \setminus B_m)} < \epsilon$. Denote by ω_n the volume of the unit ball B_1 in \mathbb{R}^n . Take any $R > m$ so that: $\left(\frac{m}{R}\right)^{n/2} \|f\|_{L^2(\mathbb{R}^n)} \leq \epsilon$. Then:

$$\begin{aligned} R^{-n/2} \|f\|_{L^1(B_R)} &= R^{-n/2} \left(\int_{B_R \setminus B_m} |f| + \int_{B_m} |f| \right) \\ &\leq R^{-n/2} |B_R|^{1/2} \epsilon + R^{-n/2} |B_m|^{1/2} \|f\|_{L^2(\mathbb{R}^n)} \\ &\leq \omega_n^{1/2} \epsilon + \left(\frac{m}{R}\right)^{n/2} \omega_n^{1/2} \|f\|_{L^2(\mathbb{R}^n)} \leq 2\omega_n^{1/2} \epsilon, \end{aligned}$$

which achieves the proof. \blacksquare

Proof of Theorem 1.1

1. Without loss of generality we may assume that $0 \in \Omega$. Let $u \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^n)$ with $\|D(u)\|_{L^2(\mathbb{R}^n)} = 1$. Define the sequence $u_k \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^n)$ by: $u_k(x) = k^{n/2-1} u(kx)$. One has:

$$\|\nabla u_k\|_{L^2(\mathbb{R}^n)} = \|\nabla u\|_{L^2(\mathbb{R}^n)}, \quad \|D(u_k)\|_{L^2(\mathbb{R}^n)} = \|D(u)\|_{L^2(\mathbb{R}^n)} = 1.$$

Let now $\phi \in \mathcal{C}_c^\infty(\Omega)$ be a nonnegative function, equal identically to 1 in a neighborhood of 0, and define: $v_k = \phi u_k$. Clearly $v_k \in W_0^{1,2}(\Omega, \mathbb{R}^n)$ and:

$$\nabla v_k = \phi \nabla u_k + u_k \otimes \nabla \phi.$$

We claim that:

$$(3.1) \quad \lim_{k \rightarrow \infty} \|\nabla v_k\|_{L^1(\Omega)} = 0,$$

$$(3.2) \quad \lim_{k \rightarrow \infty} \|\nabla v_k\|_{L^2(\Omega)} = \|\nabla u\|_{L^2(\mathbb{R}^n)},$$

$$(3.3) \quad \lim_{k \rightarrow \infty} \|D(v_k)\|_{L^2(\Omega)} = \|D(u)\|_{L^2(\mathbb{R}^n)} = 1.$$

To prove the claim, notice first that:

$$\lim_{k \rightarrow \infty} \|u_k \otimes \nabla \phi\|_{L^2(\Omega)} \leq \lim_{k \rightarrow \infty} \|\nabla \phi\|_{L^\infty} k^{-1} \|u\|_{L^2(\mathbb{R}^n)} = 0.$$

On the other hand, for all $i, j : 1 \dots n$:

$$\lim_{k \rightarrow \infty} \left\| \phi \frac{\partial}{\partial x_i} u_k^j \right\|_{L^2(\Omega)}^2 = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \left| \phi(x/k) \frac{\partial}{\partial x_i} u^j(x) \right|^2 dx = \left\| \frac{\partial}{\partial x_i} u^j \right\|_{L^2(\Omega)}^2.$$

Thus we obtain (3.2) and (3.3). Similarly:

$$\lim_{k \rightarrow \infty} \|\phi \nabla u_k\|_{L^1(\Omega)} \leq \lim_{k \rightarrow \infty} \|\phi\|_{L^\infty} k^{-n/2} \|\nabla u\|_{L^1(k\Omega)} = 0,$$

where the last equality follows by Proposition 3.1. Hence we conclude (3.1) as well.

2. Notice that by Lemma 2.1:

$$\begin{aligned} \min_{A \in L_\Omega} \|\nabla v_k - A\|_{L^2(\Omega)} &= \left\| \nabla v_k - \mathbb{P}_{L_\Omega} \oint_\Omega \nabla v_k \right\|_{L^2(\Omega)} \\ &\geq \|\nabla v_k\|_{L^2(\Omega)} - \left\| \mathbb{P}_{L_\Omega} \oint_\Omega \nabla v_k \right\|_{L^2(\Omega)} \geq \|\nabla v_k\|_{L^2(\Omega)} - |\Omega|^{-1/2} \|\nabla v_k\|_{L^1(\Omega)}. \end{aligned}$$

Now, by (3.1) and (3.2), the right hand side of the above inequality converges to $\|\nabla u\|_{L^2(\mathbb{R}^n)}$ as $k \rightarrow \infty$. On the other hand, by (1.2) and (1.3), the left hand side is bounded by $\kappa(\Omega) \|D(v_k)\|_{L^2(\Omega)}$. Therefore, passing to the limit and using (3.3), we obtain:

$$\|\nabla u\|_{L^2(\mathbb{R}^n)} \leq \kappa(\Omega) \|D(u)\|_{L^2(\mathbb{R}^n)} = \kappa(\Omega).$$

Recalling the definition (1.4) the theorem follows. ■

Example 3.2. We now show that $\kappa(Q) = \sqrt{2}$ for $Q = [0, 1]^2 \subset \mathbb{R}^2$.

Firstly, observe that (see Theorem 9.4 [12]) $L_\Omega \neq \{0\}$ if and only if Ω has a rotational symmetry. When this is not the case, then:

$$(3.4) \quad \kappa(\Omega) = \sup \left\{ \frac{\|\nabla u\|_{L^2(\Omega)}}{\|D(u)\|_{L^2(\Omega)}}; u \in W^{1,2}(\Omega, \mathbb{R}^n), u \cdot \vec{n} = 0 \text{ on } \partial\Omega \right\}.$$

In view of (3.4) and Theorem 1.1, it is hence enough to prove that for every $u \in W^{1,2}(Q, \mathbb{R}^2)$ satisfying $u^1(0, x_2) = u^1(1, x_2) = 0$ and $u^2(x_2, 0) = u^2(x_1, 0) = 0$ for all $x_1, x_2 \in [0, 1]$, there holds:

$$(3.5) \quad \int_Q |\nabla u|^2 \leq 2 \int_Q |D(u)|^2.$$

Consider first a regular vector field $u \in \mathcal{C}^2(\bar{Q}, \mathbb{R}^2)$. As in (2.2), we obtain:

$$(3.6) \quad \int_Q |D(u)|^2 = \frac{1}{2} \int_Q |\nabla u|^2 + \frac{1}{2} \int_Q |\operatorname{div} u|^2 + \int_Q (\partial_1 u^2 \partial_2 u^1 - \partial_1 u^1 \partial_2 u^2).$$

Note that:

$$\int_Q (\partial_1 u^2 \partial_2 u^1 - \partial_1 u^1 \partial_2 u^2) = \int_Q \partial_1 (u^2 \partial_2 u^1) - \int_Q \partial_2 (u^2 \partial_1 u^1),$$

and that both terms in the right hand side of the above equality integrate to 0 on Q , because of the assumed boundary condition. Thus, (3.6) yields (3.5) for $u \in \mathcal{C}^2$.

It now suffices to check that every $u \in W^{1,2}(Q, \mathbb{R}^2)$ with $u \cdot \vec{n} = 0$ on ∂Q , can be approximated by a sequence of $\mathcal{C}^2(\bar{Q}, \mathbb{R}^2)$ vector fields satisfying the same

boundary condition. To this end, define the extension $\bar{u}^1 \in W^{1,2}([0, 1] \times [-1, 2], \mathbb{R})$ of the component $u^1 \in W^{1,2}(Q, \mathbb{R})$, by:

$$\forall x_1 \in [0, 1] \quad \forall x_2 \in [-1, 2] \quad \bar{u}^1(x) = \begin{cases} u^1(x) & \text{if } x_2 \in [0, 1] \\ u^1(x_1, -x_2) & \text{if } x_2 \in [-1, 0] \\ u^1(x_1, 2 - x_2) & \text{if } x_2 \in [1, 2]. \end{cases}$$

Let $\phi : (-1, 2) \rightarrow \mathbb{R}$ be a nonnegative, smooth and compactly supported function, equal to 1 on $[0, 1]$. Then $\phi \bar{u}^1 \in W_0^{1,2}([0, 1] \times [-1, 2], \mathbb{R})$, and thus $\phi \bar{u}^1$ can be approximated in $W^{1,2}$ by a sequence $u_k^1 \in \mathcal{C}_c^\infty([0, 1] \times [-1, 2], \mathbb{R})$. Clearly, u_k^1 converges to u^1 on Q , and each $u_k^1(x) = 0$ whenever $x_1 \in \{0, 1\}$.

In a similar manner, we construct smooth approximating sequence $\{u_k^2\}_{k \geq 1}$. Writing $u_k = (u_k^1, u_k^2) \in \mathcal{C}^\infty(\bar{Q}, \mathbb{R}^2)$, we obtain the desired approximations of u . ■

Example 3.3. We now recall the construction [12] of a family of domains $\Omega^h \subset \mathbb{R}^n$ parametrised by $0 < h \ll 1$, with the property that:

$$\kappa(\Omega^h) \rightarrow \infty \quad \text{as } h \rightarrow 0.$$

Let S denote the $(n-1)$ -dimensional unit sphere in \mathbb{R}^n and let $g : S \rightarrow (0, \frac{1}{3})$ be a smooth function on S . Define:

$$\Omega^h = \left\{ (1+t)x; x \in S, t \in (hg(x) - h, hg(x)) \right\}.$$

Clearly, we may request from function g to be such that no Ω^h has any rotational symmetry, and hence $L_{\Omega^h} = \{0\}$ implies (3.4) for all h .

Let now $v : S \rightarrow \mathbb{R}^n$ be a tangent vector field given by a rotation: $v(x) = a \times x$, for some $a \in \mathbb{R}^n$. Define $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^n)$:

$$u^h(x+tx) = \left((1+t)\text{Id} + hx \otimes \nabla g(x) \right) v(x) = (1+t)(a \times x) + \langle a, x \times \nabla g(x) \rangle x.$$

One can check that u^h is tangent at $\partial\Omega^h$ and that:

$$\|\nabla u^h\|_{L^2(\Omega^h)} \geq Ch^{1/2}, \quad \|D(u^h)\|_{L^2(\Omega^h)} \leq Ch^{3/2}.$$

Hence we conclude the blow-up of Korn's constant: $\kappa(\Omega^h) \geq Ch^{-1}$ in the vanishing thickness $h \rightarrow 0$. ■

4. THE OPTIMAL KORN CONSTANT $\kappa(\Omega)$: PROOFS OF THEOREMS 1.2, 1.3 AND 1.4

Proof of Theorem 1.2

1. From (ii) and (iii) we see that the sequences:

$$\{|\nabla u_k|^2 \chi_\Omega \, dx\}_{k=1}^\infty \quad \text{and} \quad \{|D(u_k)|^2 \chi_\Omega \, dx\}_{k=1}^\infty$$

are bounded in the space of Radon measures $\mathcal{M}(\mathbb{R}^n)$. Therefore (possibly passing to subsequences), they converge weakly in $\mathcal{M}(\mathbb{R}^n)$ to some μ, ν , concentrated on $\bar{\Omega}$. That is:

$$(4.1) \quad \begin{aligned} \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \quad & \lim_{k \rightarrow \infty} \int_\Omega \phi^2 |\nabla u_k|^2 \, dx = \int_{\mathbb{R}^n} \phi^2 \, d\mu, \\ & \lim_{k \rightarrow \infty} \int_\Omega \phi^2 |D(u_k)|^2 \, dx = \int_{\mathbb{R}^n} \phi^2 \, d\nu. \end{aligned}$$

In particular, one has:

$$(4.2) \quad \mu(\bar{\Omega}) = \kappa(\Omega)^2, \quad \nu(\bar{\Omega}) = 1.$$

We now assume that:

$$(4.3) \quad \kappa(\Omega) > \kappa(\mathbb{R}^n),$$

and derive a contradiction. We will distinguish two cases: when $\mu(\Omega) > 0$ and $\mu(\Omega) = 0$.

2. First, notice that:

$$(4.4) \quad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \quad \int_{\mathbb{R}^n} \phi^2 \, d\mu \leq \kappa(\Omega)^2 \int_{\mathbb{R}^n} \phi^2 \, d\nu.$$

Indeed, for a given ϕ as above consider the sequence $v_k = \phi u_k \in W^{1,2}(\Omega, \mathbb{R}^n)$. Clearly $v_k \cdot \vec{n} = 0$ on $\partial\Omega$ and by Lemma 2.1 we have:

$$(4.5) \quad \left\| \nabla v_k - \mathbb{P}_{L\Omega} \int_{\Omega} \nabla v_k \right\|_{L^2(\Omega)} \leq \kappa(\Omega) \|D(v_k)\|_{L^2(\Omega)}.$$

Since $\nabla v_k = \phi \nabla u_k + u_k \otimes \nabla \phi$, the sequence $\int_{\Omega} \nabla v_k$ converges to 0 in $\mathbb{R}^{n \times n}$ by (i). The same convergence must be true for the respective sequence of projections. Similarly, $\lim_{k \rightarrow \infty} \|u_k \otimes \nabla \phi\|_{L^2(\Omega)} = 0$ by (i). Hence (4.5), after passing to the limit with $k \rightarrow \infty$ yields:

$$\lim_{k \rightarrow \infty} \|\phi \nabla u_k\|_{L^2(\Omega)} \leq \kappa(\Omega) \lim_{k \rightarrow \infty} \|\phi D(u_k)\|_{L^2(\Omega)},$$

which in view of (4.1) proves (4.4).

Assume now that $\mu(\Omega) > 0$. In this case we are ready to derive a contradiction. Let B be an open ball, compactly contained in Ω , with $\mu(B) > 0$. By (4.4):

$$(4.6) \quad \mu(\bar{\Omega} \setminus B) \leq \kappa(\Omega)^2 \nu(\bar{\Omega} \setminus B).$$

On the other hand, recalling the definition (1.4) and reasoning exactly as in the proof of (4.4), we get:

$$\forall \phi \in \mathcal{C}_c^\infty(B) \quad \int_B \phi^2 \, d\mu \leq \kappa(\mathbb{R}^n)^2 \int_B \phi^2 \, d\nu,$$

which implies:

$$(4.7) \quad \mu(B) \leq \kappa(\mathbb{R}^n)^2 \nu(B).$$

Now, both sides of (4.7) are positive, so by (4.3): $\mu(B) < \kappa(\Omega)^2 \nu(B)$. Together with (4.6) this yields:

$$\mu(\bar{\Omega}) < \kappa(\Omega)^2 \nu(\bar{\Omega}),$$

contradicting (4.2).

3. It remains to consider the case $\mu(\Omega) = 0$, when the measure μ concentrates on $\partial\Omega$, due to the lack of the equiintegrability of the sequence $\{|\nabla u_k|^2\}_{k=1}^\infty$ close to $\partial\Omega$. We will prove that:

$$(4.8) \quad \mu(\partial\Omega) \leq \kappa(\mathbb{R}^n)^2 \nu(\partial\Omega).$$

Both sides of (4.8) are positive, and so (4.3) in view of the assumption $\mu(\Omega) = 0$ implies:

$$\mu(\bar{\Omega}) = \mu(\partial\Omega) < \kappa(\Omega)^2 \nu(\partial\Omega) \leq \kappa(\Omega)^2 \nu(\bar{\Omega}),$$

contradicting (4.2). This will end the proof of the theorem.

Towards establishing (4.8), let $\theta : [0, \infty) \rightarrow [0, 1]$ be a smooth, non-increasing function such that:

$$\theta(t) = 1 \text{ for } t \in [0, 1], \quad \theta(t) = 0 \text{ for } t \geq 2.$$

Define: $\phi_k(x) = \theta(k \text{dist}(x, \partial\Omega))$. For large k we have $\phi_k \in \mathcal{C}_c^\infty((\partial\Omega)_\epsilon)$ on a small open neighborhood $(\partial\Omega)_\epsilon$ of $\partial\Omega$. By (4.1), for some increasing sequence $\{n_k\}_{k=1}^\infty$:

$$(4.9) \quad \mu(\partial\Omega) = \lim_{k \rightarrow \infty} \|\phi_k \nabla u_{n_k}\|_{L^2(\Omega)}^2, \quad \nu(\partial\Omega) = \lim_{k \rightarrow \infty} \|\phi_k D(u_{n_k})\|_{L^2(\Omega)}^2.$$

To simplify the notation, we will pass to subsequences and write $n_k = k$.

Define the extension of u_k on $(\partial\Omega)_\epsilon$ by reflecting the normal components oddly and tangential components evenly, across $\partial\Omega$. That is, denoting by $\pi : (\partial\Omega)_\epsilon \rightarrow \partial\Omega$ the projection onto $\partial\Omega$ along the normal vectors \vec{n} , so that:

$$(x - \pi(x)) \cdot \vec{n}(\pi(x)) = 0 \quad \forall x \in (\partial\Omega)_\epsilon,$$

let, for all $x \in (\partial\Omega)_\epsilon \setminus \Omega$:

$$(4.10) \quad \begin{aligned} u_k(x) \cdot \vec{n}(\pi(x)) &= -u_k(2\pi(x) - x) \cdot \vec{n}(\pi(x)), \\ u_k(x) \cdot \tau &= u_k(2\pi(x) - x) \cdot \tau \quad \forall \tau \in T_{\pi(x)}\partial\Omega. \end{aligned}$$

Since $u_k \cdot \vec{n} = 0$ on $\partial\Omega$, the above defined extension u_k is still $W^{1,2}$ regular. By (1.4) there holds:

$$\|\nabla(\phi_k u_k)\|_{L^2(\mathbb{R}^n)} \leq \kappa(\mathbb{R}^n) \|D(\phi_k u_k)\|_{L^2(\mathbb{R}^n)}.$$

Again, by taking $\{n_k\}$ in (4.9) converging to ∞ sufficiently fast, we may without loss of generality assume that $\|u_k\|_{L^2(\Omega)} \leq 1/k^2$. Therefore:

$$(4.11) \quad \lim_{k \rightarrow \infty} \|\phi_k \nabla u_k\|_{L^2(\mathbb{R}^n)} \leq \kappa(\mathbb{R}^n) \lim_{k \rightarrow \infty} \|\phi_k D(u_k)\|_{L^2(\mathbb{R}^n)}.$$

Consider the quantity:

$$I = \lim_{k \rightarrow \infty} \left\{ \int_{\mathbb{R}^n \setminus \Omega} |\phi_k \nabla u_k|^2 - \int_{\Omega} |\phi_k \nabla u_k|^2 \right\}.$$

After changing the variables in the first integral and noting that:

$$\det \nabla(2\pi(x) - x) = \det(2\nabla\pi(x) - \text{Id}) = -1 + \mathcal{O}(1)|x - \pi(x)|,$$

we obtain:

$$(4.12) \quad \begin{aligned} I &= \lim_{k \rightarrow \infty} \int_{\Omega} \left\{ |\phi_k(x) \nabla u_k(2\pi(x) - x)|^2 - |\phi_k(x) \nabla u_k(x)|^2 \right\} dx \\ &= \lim_{k \rightarrow \infty} \int_{\Omega \cap \{\text{dist}(x, \partial\Omega) < 1/k\}} \left\{ |\nabla u_k(2\pi(x) - x)|^2 - |\nabla u_k(x)|^2 \right\} dx. \end{aligned}$$

The definition of extension (4.10) yields now the following identities, for each $x \in (\partial\Omega)_\epsilon$ and each $\tau, \eta \in T_{\pi(x)}\partial\Omega$:

$$(4.13) \quad \begin{aligned} \partial_\tau(u_k \cdot \eta)(2\pi(x) - x) &= \left(1 + \mathcal{O}(1)|x - \pi(x)|\right) \partial_\tau(u_k \cdot \eta)(x), \\ \partial_{\vec{n}(\pi(x))}(u_k \cdot \eta)(2\pi(x) - x) &= -\partial_{\vec{n}(\pi(x))}(u_k \cdot \eta)(x), \\ \partial_\tau(u_k \cdot \vec{n}(\pi(x)))(2\pi(x) - x) &= \left(-1 + \mathcal{O}(1)|x - \pi(x)|\right) \partial_\tau(u_k \cdot \vec{n}(\pi(x)))(x), \\ \partial_{\vec{n}(\pi(x))}(u_k \cdot \vec{n}(\pi(x)))(2\pi(x) - x) &= \partial_{\vec{n}(\pi(x))}(u_k \cdot \vec{n}(\pi(x)))(x). \end{aligned}$$

Since $\eta \partial_\tau v_k = \partial_\tau(v_k \eta) - v_k \partial_\tau \eta$, we see that equating the contribution of all components in (4.12) and recalling (iii) we have:

$$I = 0.$$

In the same manner, (4.13) implies that $|D(u_k)(2\pi(x) - x)|^2$ equals to $|D(u_k)(x)|^2$ plus lower order terms whose integrals on $\Omega \cap \{\text{dist}(x, \partial\Omega) < 1/k\}$ vanish, as $k \rightarrow \infty$. Hence also:

$$II = \lim_{k \rightarrow \infty} \left\{ \int_{\mathbb{R}^n \setminus \Omega} |\phi_k D(u_k)|^2 - \int_{\Omega} |\phi_k D(u_k)|^2 \right\} = 0.$$

Therefore:

$$(4.14) \quad \begin{aligned} \lim_{k \rightarrow \infty} \|\phi_k \nabla u_k\|_{L^2(\mathbb{R}^n)} &= 2 \lim_{k \rightarrow \infty} \|\phi_k \nabla u_k\|_{L^2(\Omega)}, \\ \lim_{k \rightarrow \infty} \|\phi_k D(u_k)\|_{L^2(\mathbb{R}^n)} &= 2 \lim_{k \rightarrow \infty} \|\phi_k D(u_k)\|_{L^2(\Omega)}. \end{aligned}$$

Combining (4.14), (4.11) with (4.9) proves (4.8). ■

Proof of Theorem 1.3

It is enough to assume that $A_0 = 0$. Let $\{u_k\}_{k=1}^\infty$ be a maximizing sequence of (1.3), that is: $u_k \in W^{1,2}(\Omega, \mathbb{R}^n)$, $u_k \cdot \vec{n} = 0$ on $\partial\Omega$, $\|D(u_k)\|_{L^2(\Omega)} = 1$ and $\lim_{k \rightarrow \infty} \|\nabla u_k - \mathbb{P}_{L_\Omega} \oint_\Omega \nabla u_k\|_{L^2(\Omega)} = \kappa(\Omega)$.

By modifying u_k we may, without loss of generality, assume that:

$$(4.15) \quad \mathbb{P}_{L_\Omega} \oint_\Omega \nabla u_k = 0, \quad \lim_{k \rightarrow \infty} \|\nabla u_k\|_{L^2(\Omega)} = \kappa(\Omega).$$

Using Lemma 2.3 (after possibly passing to a subsequence), we have:

$$(4.16) \quad u_k \rightharpoonup u \quad \text{weakly in } W^{1,2}(\Omega, \mathbb{R}^n),$$

for some u satisfying $u \cdot \vec{n} = 0$ on $\partial\Omega$.

We now show that (1.6) holds with $A_0 = 0$. First of all, by applying Theorem 1.2 to the sequence $\{u_k\}$, we see that $u \neq 0$. Further, (4.16) implies that $\mathbb{P}_{L_\Omega} \oint_\Omega \nabla u = \lim_{k \rightarrow \infty} \mathbb{P}_{L_\Omega} \oint_\Omega \nabla u_k = 0$, so:

$$(4.17) \quad \|\nabla u\|_{L^2(\Omega)} \leq \kappa(\Omega) \|D(u)\|_{L^2(\Omega)}.$$

Since $\mathbb{P}_{L_\Omega} \oint_\Omega \nabla(u_k - u) = 0$, there also holds:

$$\|\nabla(u_k - u)\|_{L^2(\Omega)} \leq \kappa(\Omega) \|D(u_k - u)\|_{L^2(\Omega)}.$$

Squaring both sides of the above inequality, passing to the limit with $k \rightarrow \infty$ and recalling (4.15) and (4.16), we obtain:

$$\kappa(\Omega)^2 - \|\nabla u\|_{L^2(\Omega)}^2 \leq \kappa(\Omega)^2 \left(1 - \|D(u)\|_{L^2(\Omega)}^2\right).$$

Together with (4.17) this proves:

$$\|\nabla u\|_{L^2(\Omega)} = \kappa(\Omega) \|D(u)\|_{L^2(\Omega)},$$

yielding the result. ■

Proof of Theorem 1.4

1. Let E be the set in (1.7). It is clear that $u \in E$ implies $\lambda u \in E$, for all $\lambda \in \mathbb{R}$. If $u_1, u_2 \in E$, then by Lemma 2.1:

$$(4.18) \quad \left\| \nabla u_i - \mathbb{P}_{L_\Omega} \oint_\Omega \nabla u_i \right\|_{L^2(\Omega)} = \kappa(\Omega) \|D(u_i)\|_{L^2(\Omega)} \quad \forall i = 1, 2.$$

On the other hand, by the linearity of the operator \mathbb{P}_{L_Ω} and by (1.2), (1.3):
(4.19)

$$\left\| \nabla(u_1 \pm u_2) - \left(\mathbb{P}_{L_\Omega} \int_{\Omega} \nabla u_1 \pm \mathbb{P}_{L_\Omega} \int_{\Omega} \nabla u_2 \right) \right\|_{L^2(\Omega)} \leq \kappa(\Omega) \|D(u_1 \pm u_2)\|_{L^2(\Omega)}.$$

Squaring the two inequalities in (4.19) and equating the terms from (4.18) we obtain:

$$\left\langle \nabla u_1 - \mathbb{P}_{L_\Omega} \int_{\Omega} \nabla u_1, \nabla u_2 - \mathbb{P}_{L_\Omega} \int_{\Omega} \nabla u_2 \right\rangle_{L^2(\Omega)} = \kappa(\Omega)^2 \langle D(u_1), D(u_2) \rangle_{L^2(\Omega)}.$$

Therefore, (4.19) is actually true as the equality. We hence conclude that $u_1 + u_2 \in E$, proving that E is a linear space.

The closedness of E follows by noting that if a sequence u_k converges to u in $W^{1,2}(\Omega, \mathbb{R}^n)$ then the minimizing matrices $\mathbb{P}_{L_\Omega} \int \nabla u_k$ converge to $\mathbb{P}_{L_\Omega} \int \nabla u$.

2. To prove the second claim, we argue by contradiction. Assume that the space E is of infinite dimension. Then it admits a Hilbertian (orthonormal in $W^{1,2}(\Omega, \mathbb{R}^n)$) basis $\{u_k\}_{k=1}^\infty$. It is easy to see that there must be:

$$(4.20) \quad u_k \rightharpoonup 0 \quad \text{weakly in } W^{1,2}(\Omega, \mathbb{R}^n).$$

We now notice that:

$$(4.21) \quad \liminf_{k \rightarrow \infty} \|D(u_k)\|_{L^2(\Omega)} > 0.$$

Because otherwise, by Korn's inequality (1.2) there would be:

$$\liminf_{k \rightarrow \infty} \left\| \nabla u_k - \mathbb{P}_{L_\Omega} \int_{\Omega} \nabla u_k \right\|_{L^2(\Omega)} = 0,$$

and since by (4.20) $\lim_{k \rightarrow \infty} \int \nabla u_k = 0$, there follows that $\liminf_{k \rightarrow \infty} \|\nabla u_k\|_{L^2(\Omega)} = 0$. In view of the Poincaré inequality (2.3), we hence obtain $\liminf_{k \rightarrow \infty} \|u_k\|_{W^{1,2}(\Omega)} = 0$, in contradiction with the orthonormality of the sequence $\{u_k\}_{k=1}^\infty$.

Define: $v_k = u_k / \|D(u_k)\|_{L^2(\Omega)}$. Clearly, there holds:

$$\|D(v_k)\|_{L^2(\Omega)} = 1, \quad \|\nabla v_k\|_{L^2(\Omega)} = \kappa(\Omega),$$

and because of (4.21) we also have: $v_k \rightharpoonup 0$ weakly in $W^{1,2}(\Omega, \mathbb{R}^n)$. By Theorem 1.2 there follows $\kappa(\Omega) = \kappa(\mathbb{R}^n) = \sqrt{2}$, which is a desired contradiction. \blacksquare

5. THE OPTIMAL GEOMETRIC RIGIDITY CONSTANT IN \mathbb{R}^2

To prove Theorems 1.5, 1.6, 1.7 we need some preliminary discussion.

Lemma 5.1. *Assume that $w \in L^2_{loc}(\mathbb{R}^n)$ and $\Delta w = 0$. If $w = f + g$ with $f \in L^2(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$, then $w \equiv \text{const}$.*

Proof. Fix $x_0, y_0 \in \mathbb{R}^n$. For any $r > 0$ we have:

$$\begin{aligned} |w(x_0) - w(y_0)| &= \left| \oint_{B_r(x_0)} w - \oint_{B_r(y_0)} w \right| = \frac{1}{|B_r|} \left| \int_{B_r(x_0) \Delta B_r(y_0)} w \right| \\ &\leq \left(\frac{1}{|B_r|} \int_{B_r(x_0) \Delta B_r(y_0)} |f| \right) + \left(\frac{1}{|B_r|} \int_{B_r(x_0) \Delta B_r(y_0)} |g| \right) \\ &\leq \frac{|B_r(x_0) \Delta B_r(y_0)|^{1/2}}{|B_r|} \|f\|_{L^2} + \frac{|B_r(x_0) \Delta B_r(y_0)|}{|B_r|} \|g\|_{L^\infty} \\ &\leq \left(\frac{1}{|B_r|} + \frac{|B_r(x_0) \Delta B_r(y_0)|}{|B_r|} \right) (\|f\|_{L^2} + \|g\|_{L^\infty}), \end{aligned}$$

where by Δ we denote the symmetric difference of two sets: $B_1 \Delta B_2 = (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$. The quantity in the first parentheses above clearly converges to 0 as $r \rightarrow \infty$. Therefore $w(x_0) = w(y_0)$, which achieves the proof. \blacksquare

Lemma 5.2. *Let $f \in L^2(\mathbb{R}^2, \mathbb{R}^{2 \times 2})$ and let $u \in W_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ satisfy:*

$$(5.1) \quad -\Delta u = \operatorname{div} f \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

Then u can be decoupled as:

$$(5.2) \quad u = v + w; \quad v, w \in L_{loc}^2, \quad \nabla v \in L^2, \quad \nabla w \in L_{loc}^2, \quad -\Delta w = 0 \quad \text{in } \mathbb{R}^2.$$

Moreover:

$$(5.3) \quad \nabla v = \lim_{m \rightarrow \infty} \nabla v_m \quad \text{strongly in } L^2(\mathbb{R}^n), \quad \text{for some } v_m \in \mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{R}^2).$$

Proof. For each $m \in \mathbb{N}$, let v_m be the solution to:

$$(5.4) \quad \begin{cases} v_m \in W_0^{1,2}(B_m) \\ -\int_{\mathbb{R}^2} \nabla v_m : \nabla \phi = \int_{\mathbb{R}^2} f : \nabla \phi \quad \forall \phi \in W_0^{1,2}(B_m), \end{cases}$$

whose existence and uniqueness follow from the Lax-Milgram theorem, together with:

$$\|\nabla v_m\|_{L^2} \leq \|f\|_{L^2(B_m)} \leq \|f\|_{L^2(\mathbb{R}^2)}.$$

Therefore, passing to a subsequence:

$$(5.5) \quad \nabla v_m \rightharpoonup z \quad \text{weakly in } L^2(\mathbb{R}^2)$$

and also :

$$(5.6) \quad \operatorname{curl} z = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

Condition (5.6) is now equivalent to: $z = \nabla v$. This can be seen, for example, via Helmholtz decomposition [6]:

$$z = z_0 + \nabla v; \quad v \in L_{loc}^2, \quad \nabla v, z_0 \in L^2, \quad \operatorname{div} z_0 = 0 \quad \text{in } \mathcal{D}'.$$

Since from (5.6) also $\operatorname{curl} z_0 = 0$, hence the components of z_0 satisfy the Cauchy-Riemann equations, and therefore $\Delta z_0 = 0$. Recalling that $z_0 \in L^2(\mathbb{R}^2)$ it follows by Lemma 5.1 that $z_0 = 0$. Consequently, by (5.5):

$$(5.7) \quad \nabla v_m \rightharpoonup \nabla v \quad \text{weakly in } L^2(\mathbb{R}^2).$$

Passing to the limit in (5.4), we obtain: $-\Delta v = \operatorname{div} f$ in \mathcal{D}' , hence $-\Delta w = 0$, for $w = u - v$ and (5.2) is proven.

Finally, by Mazur's theorem and (5.7), ∇v is the strong L^2 -limit of $\nabla \tilde{v}_m$ which are gradients of some finite (in fact, convex) linear combinations \tilde{v}_m of v_m . Clearly, each $\tilde{v}_m \in W_0^{1,2}(B_{r_m})$ and the result in (5.3) follows by density of $\mathcal{C}_c^\infty(B_{r_m})$ in $W_0^{1,2}(B_{r_m})$. \blacksquare

Remark 5.3. Note that one can directly show that ∇v_m in Lemma 5.2 converges strongly in $L^2(\mathbb{R}^2)$. Let $k > m$. Extending v_m by zero to \mathbb{R}^2 , so that $v_m \in W_0^{1,2}(B_k)$, and taking $\phi = v_m$ in the equation (5.4) written for v_k , we get:

$$\int_{B_k} \nabla v_k : \nabla v_m = - \int_{B_k} f : \nabla v_m = - \int_{B_m} f : \nabla v_m = \int_{B_m} |\nabla v_m|^2.$$

The last equality above follows by taking $\phi = v_m$ in the equation (5.4) written for v_m . Now, passing $m \rightarrow \infty$ implies, by the weak convergence in (5.5):

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}^2} |\nabla v_m|^2 = \int_{\mathbb{R}^2} \nabla v_k : \nabla v.$$

Finally, passing $k \rightarrow \infty$ yields:

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}^2} |\nabla v_m|^2 = \int_{\mathbb{R}^2} |\nabla v|^2.$$

The claim (5.3) now follows, since convergence of norms in presence of the weak convergence implies strong convergence in $L^2(\mathbb{R}^2)$.

Lemma 5.4. *Let $u \in W_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ and $\nabla u \in L^2(\mathbb{R}^2)$. Then:*

$$(5.8) \quad \int_{\mathbb{R}^2} \det \nabla u = 0.$$

Proof. Since $\Delta u = \operatorname{div} \nabla u$ in $\mathcal{D}'(\mathbb{R}^2)$ we may apply Lemma 5.2 to $f = \nabla u \in L^2(\mathbb{R}^2)$ and write $u = v + w$ satisfying (5.2). Since $\Delta w = 0$ and $\nabla w = \nabla u - \nabla v \in L^2(\mathbb{R}^2)$, it follows from Lemma 5.1 that $\nabla w = 0$ and hence by (5.3):

$$\nabla u = \nabla v = \lim_{m \rightarrow \infty} \nabla v_m \quad \text{strongly in } L^2(\mathbb{R}^2), \text{ for some } v_m \in \mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{R}^2).$$

It remains to prove (5.8) for $u \in \mathcal{C}_c^\infty$, which is a standard argument. Let $\operatorname{supp} u \subset B_r$. We have:

$$\begin{aligned} \int_{\mathbb{R}^2} \det \nabla u &= \int_{B_r} (\partial_1 u^1 \partial_2 u^2 - \partial_1 u^2 \partial_2 u^1) \\ &= \int_{B_r} (\partial_1 (u^1 \partial_2 u^2) - \partial_2 (u^1 \partial_1 u^2)) = \int_{\partial B_r} (u^1 \partial_2 u^2, u^1 \partial_1 u^2) \vec{n} = 0, \end{aligned}$$

where we used integration by parts and the divergence theorem. \blacksquare

We finally need to recall the conformal–anticonformal decomposition of 2×2 matrices. Let $\mathbb{R}_c^{2 \times 2}$ and $\mathbb{R}_a^{2 \times 2}$ denote, respectively, the spaces of conformal and anticonformal matrices:

$$\mathbb{R}_c^{2 \times 2} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix}; a, b \in \mathbb{R} \right\}, \quad \mathbb{R}_a^{2 \times 2} = \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix}; a, b \in \mathbb{R} \right\}.$$

It is easy to see that $\mathbb{R}^{2 \times 2} = \mathbb{R}_c^{2 \times 2} \oplus \mathbb{R}_a^{2 \times 2}$ because both spaces have dimension 2 and they are mutually orthogonal: $A : B = 0$ for all $A \in \mathbb{R}_c^{2 \times 2}$ and $B \in \mathbb{R}_a^{2 \times 2}$.

For $F = [F_{ij}]_{i,j:1,2} \in \mathbb{R}^{2 \times 2}$, its projections F^c on $\mathbb{R}_c^{2 \times 2}$, and F^a on $\mathbb{R}_a^{2 \times 2}$ are:

$$F^c = \frac{1}{2} \begin{bmatrix} F_{11} + F_{22} & F_{12} - F_{21} \\ F_{21} - F_{12} & F_{11} + F_{22} \end{bmatrix}, \quad F^a = \frac{1}{2} \begin{bmatrix} F_{11} - F_{22} & F_{12} + F_{21} \\ F_{12} + F_{21} & F_{22} - F_{11} \end{bmatrix}.$$

It follows that:

$$(5.9) \quad F = F^c + F^a \quad \text{and} \quad |F|^2 = |F^c|^2 + |F^a|^2$$

and, by a direct calculation:

$$(5.10) \quad \det F = 2(|F^c|^2 - |F^a|^2).$$

Since $SO(2) = \left\{ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \theta \in [0, 2\pi) \right\} \subset \mathbb{R}_c^{2 \times 2}$, it also follows that:

$$(5.11) \quad \text{dist}(F, SO(2)) \geq \text{dist}(F, \mathbb{R}_c^{2 \times 2}) = |F^a|$$

which implies:

$$(5.12) \quad |\text{cof } F - F| = |-2F^a| \leq 2\text{dist}(F, SO(2)).$$

Finally, recall that the cofactor matrix in dimension 2 is given by:

$$\text{cof } F = \begin{bmatrix} F_{22} & -F_{21} \\ -F_{12} & F_{11} \end{bmatrix}.$$

We now state the following first result towards proving Theorem 1.5.

Lemma 5.5. *Let $u \in W_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ and assume that $\text{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2, \mathbb{R})$. Then there exists $R_0 \in SO(2)$ such that:*

$$\int_{\mathbb{R}^2} |\nabla u(x) - R_0|^2 \, dx \leq 2 \int_{\mathbb{R}^2} \text{dist}^2(\nabla u(x), SO(2)) \, dx.$$

Proof. From the assumption $\text{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2)$ and (5.12) we deduce:

$$f := \text{cof } \nabla u - \nabla u \in L^2(\mathbb{R}^2).$$

Taking divergence of f and recalling that $\text{div } \text{cof } \nabla u = 0$ we obtain that $-\Delta u = \text{div } f$. In view of Lemma 5.2 we now write:

$$(5.13) \quad u = v + w$$

where v and w satisfy (5.2). We now prove that:

$$(5.14) \quad \nabla w \equiv R_0 \in SO(2).$$

For $\epsilon > 0$ sufficiently small, define:

$$g(x) = \begin{cases} \mathbb{P}_{SO(2)} \nabla u(x) & \text{if } \text{dist}(\nabla u(x), SO(2)) < \epsilon \\ \text{Id} & \text{otherwise} \end{cases}$$

Then:

$$(5.15) \quad \nabla w = g + h; \quad g \in L^\infty(\mathbb{R}^2) \text{ and } h \in L^2(\mathbb{R}^2).$$

The assertion $h = \nabla w - g = \nabla u - g + \nabla v \in L^2(\mathbb{R}^2)$ follows from the assumption $\text{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2)$ as follows. We already know that $\nabla v \in L^2(\mathbb{R}^2)$ by (5.13).

For $h_1 = \nabla u - g$ note that $|h_1(x)| = \text{dist}(\nabla u, SO(2))$ when $\text{dist}(\nabla u(x), SO(2)) < \epsilon$, while when $\text{dist}(\nabla u(x), SO(2)) \geq \epsilon$, we have:

$$\begin{aligned} |h_1(x)| &= |\nabla u(x) - \text{Id}| \leq \text{dist}(\nabla u(x), SO(2)) + \text{diam}(SO(2)) \\ &\leq \left(1 + \frac{\text{diam}(SO(2))}{\epsilon}\right) \text{dist}(\nabla u(x), SO(2)). \end{aligned}$$

Since ∇w is harmonic in \mathbb{R}^2 , (5.15) implies that $\nabla w \equiv R_0$ is constant by Lemma 5.1. But $\text{dist}(R_0, SO(2)) \leq \text{dist}(\nabla u, SO(2)) + |\nabla v| \in L^2(\mathbb{R}^2)$, so $R_0 \in SO(2)$ and (5.14) is now established.

By (5.14) and (5.13) we have:

$$(5.16) \quad \nabla u = \nabla v + R_0.$$

Since $\int \det \nabla v = 0$ by Lemma 5.4, we obtain by (5.10):

$$(5.17) \quad \int_{\mathbb{R}^2} |(\nabla v)^c|^2 = \int_{\mathbb{R}^2} |(\nabla v)^a|^2.$$

Consequently:

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla u - R_0|^2 &= \int_{\mathbb{R}^2} |\nabla v|^2 = \int_{\mathbb{R}^2} |(\nabla v)^c|^2 + \int_{\mathbb{R}^2} |(\nabla v)^a|^2 = 2 \int_{\mathbb{R}^2} |(\nabla v)^a|^2 \\ &= 2 \int_{\mathbb{R}^2} |(\nabla v + R_0)^a|^2 \leq 2 \int_{\mathbb{R}^2} \text{dist}^2(\nabla v + R_0, SO(2)) \\ &= 2 \int_{\mathbb{R}^2} \text{dist}^2(\nabla u, SO(2)), \end{aligned}$$

where we used (5.16), (5.9), (5.17), (5.11) and the fact that $(R_0)^a = 0$. This achieves the proof. \blacksquare

Proof of Theorem 1.6

1. Without loss of generality we may assume that $R_0 = \text{Id}$. We shall look for a function $u \in W_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ such that:

$$(5.18) \quad \nabla u(x) = R(\alpha(x)) + \begin{bmatrix} a(x) & b(x) \\ b(x) & -a(x) \end{bmatrix}, \quad \text{with } R(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix},$$

and:

$$(5.19) \quad \nabla u(x) - \text{Id} \in L^2(\mathbb{R}^2)$$

Indeed, note that by Lemma 5.4, (5.19) and (5.10):

$$\int_{\mathbb{R}^2} |(\nabla u)^c - \text{Id}|^2 = \int_{\mathbb{R}^2} |(\nabla u)^a|^2.$$

Hence, by (5.9):

$$\int_{\mathbb{R}^2} |\nabla u - \text{Id}|^2 = 2 \int_{\mathbb{R}^2} |(\nabla u)^a|^2 = 2 \int_{\mathbb{R}^2} \text{dist}^2(\nabla u, SO(2)).$$

because $(\nabla u)^c = R(\alpha) \in SO(2)$. Since $(\nabla u)^a = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ it also follows that $\int_{\mathbb{R}^2} \text{dist}^2(\nabla u, SO(2)) = 2 \int_{\mathbb{R}^2} (a^2 + b^2)$.

On the other hand, there is always the unique rotation R which makes the quantity in the left hand side of (1.10) finite:

$$\int_{\mathbb{R}^2} |\nabla u - R|^2 \geq \frac{1}{2} \int_{\mathbb{R}^2} |R - \text{Id}|^2 - \int_{\mathbb{R}^2} |\nabla u - \text{Id}|^2.$$

This proves the theorem, provided (5.18) and (5.19) hold.

2. We shall show that for any $\alpha \in L^2(\mathbb{R}^2, \mathbb{R})$ there exists a vector field $g = (a, b)^T \in L^2(\mathbb{R}^2, \mathbb{R}^2)$ satisfying (5.18). Then (5.19) will follow automatically, as:

$$\int |R(\alpha) - \text{Id}|^2 = 2 \int (\cos \alpha - 1)^2 + (\sin \alpha)^2 = 2 \int (2 - 2 \cos \alpha) \leq 2 \int |\alpha|^2.$$

The last inequality above follows by noting that the function $\alpha \mapsto \alpha^2 + 2 \cos \alpha - 2$ attains its minimum value 0 at $\alpha = 0$, since $(\alpha^2 + 2 \cos \alpha - 2)' = 2(\alpha - \sin \alpha)$ is positive for $\alpha > 0$ and negative for $\alpha < 0$.

Fix $\alpha \in L^2(\mathbb{R}^2)$. The map $u \in W_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ with ∇u of the form (1.11) exists if and only if the right hand side in (1.11) is curl-free, i.e.:

$$(5.20) \quad \begin{cases} \text{curl } g = \text{div } f \\ \text{div } g = \text{curl } f \end{cases} \quad \text{in } \mathcal{D}'(\mathbb{R}^2)$$

where:

$$f = (\sin \alpha, \cos \alpha - 1)^T \in L^2(\mathbb{R}^2, \mathbb{R}^2).$$

The system (5.20) can be solved by Fourier transform, namely:

$$(5.21) \quad g = \mathcal{F}^{-1}(h), \quad h(x) = - \left\langle \frac{x^\perp}{|x|}, \mathcal{F}(f)(x) \right\rangle \frac{x}{|x|} + \left\langle \frac{x}{|x|}, \mathcal{F}(f)(x) \right\rangle \frac{x^\perp}{|x|},$$

where $x^\perp = (-x_2, x_1)$. Here \mathcal{F} stands for the Fourier transform of $L^2(\mathbb{R}^2, \mathbb{C})$ and we identify the complex variable functions with the \mathbb{R}^2 -valued vector fields.

Note that from (5.21) it follows that:

$$(5.22) \quad \begin{aligned} \forall x \in \mathbb{R}^2 \quad & \langle \mathcal{F}(g)(x), x^\perp \rangle = \langle \mathcal{F}(f)(x), x \rangle \\ & \langle \mathcal{F}(g)(x), x \rangle = - \langle \mathcal{F}(f)(x), x^\perp \rangle \end{aligned}$$

which precisely implies (5.20). Therefore, for every $f \in L^2(\mathbb{R}^2)$ there exists a unique $g \in L^2(\mathbb{R}^2)$ solving (5.20). This achieves the proof of the theorem. Moreover:

$$\|g\|_{L^2} = \|h\|_{L^2} = \|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2},$$

by Plancherel identity and by inspecting (5.21). ■

This concludes the proof of Theorem 1.5 as well. In view of the argument in the above proof, Theorem 1.7 will follow in view of:

Lemma 5.6. *If $\int |\nabla u - R_0|^2 = 2 \int \text{dist}^2(\nabla u, SO(2)) < \infty$ then ∇u must be of the form (5.18) with $R(\alpha) - R_0 \in L^2(\mathbb{R}^2)$.*

Proof. Note that by (5.9): $|\nabla u - R|^2 = |(\nabla u)^c - R|^2 + |(\nabla u)^a|^2$ for any $R \in SO(2)$. Hence taking infimum over all rotations, we get:

$$(5.23) \quad \text{dist}^2(\nabla u, SO(2)) = \text{dist}^2((\nabla u)^c, SO(2)) + |(\nabla u)^a|^2.$$

In particular:

$$(\nabla u)^a \in L^2(\mathbb{R}^2).$$

Further, by (5.10) and Lemma 5.4:

$$\int |(\nabla u)^c - R_0|^2 = \int |(\nabla u)^a|^2.$$

Therefore, by (5.9) and (5.23):

$$\begin{aligned} \int |(\nabla u)^c - R_0|^2 &= \frac{1}{2} \int |\nabla u - R_0|^2 = \int \text{dist}^2(\nabla u, SO(2)) \\ &= \int \text{dist}^2((\nabla u)^c, SO(2)) + \int |(\nabla u)^a|^2 \\ &= \int \text{dist}^2((\nabla u)^c, SO(2)) + \int |(\nabla u)^c - R_0|^2, \end{aligned}$$

which implies that $\int \text{dist}^2((\nabla u)^c, SO(2)) = 0$ and hence: $(\nabla u(x))^c \in SO(2)$ for a.e. x . Consequently, ∇u has the form in (5.18) and:

$$R(\alpha) - R_0 = \nabla u - R_0 - (\nabla u)^a \in L^2(\mathbb{R}^2)$$

by (5.23). ■

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UNIVERSITY OF PITTSBURGH, DEPARTMENT OF MATHEMATICS, 301 THACKERAY HALL, PITTSBURGH, PA 15260, USA

E-mail address: `lewicka@pitt.edu`

INSTITUT FÜR ANGEWANDTE MATHEMATIK, UNIVERSITÄT BONN, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

E-mail address: `stefan.mueller@hcm.uni-bonn.de`